NASA TT F-474

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Translation of "K teorii kulonovskogo vzaimodeystviya."
Joint Institute for Nuclear Research, Dubna,
Report No. JINR-P-2638, 1966.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ON THE THEORY OF COULOMB INTERACTION

I. I. Tugov[1]

A differential equation in the momentum space has been obtained for a many-dimensional Coulomb problem. An equivalence is shown of this equation to the wave equation in a space of constant curvature $K>0$ for a discrete spectrum $E<0$ and $K<0$ for scattering in a field of attraction and in a field of repulsion. Solving Killing's equations gives generators of the symmetry group of the equation. Possible sets of quantum numbers are investigated. An explicit form of diagonal operators is presented and normalized systems of eigenfunctions of the discrete and continuous spectra have been constructed for a number of coordinate systems in the three-dimensional case. When scattering in a field of attraction and in a repulsion field, the wave functions are infinite equivalent representations of the Lorentz groups belonging to the principal series of unitary representations.

1. Introduction

The application of a general method for studying the group properties of /3* some second-order partial differential equations [1] to a differential equation describing an n-dimensional problem in momentum space makes it possible to study possible sets of quantum numbers and to give a method for constructing corresponding normalized systems of eigenfunctions.

In 1935, Fok showed that the Schrödinger equation for the hydrogen atom possesses the symmetry of a four-dimensional sphere in the sense that wave functions in momentum space belonging to a single level of the discrete spectrum realize an irreducible representation of the O(4) group. As shown in reference 2, the Schrödinger equation for the hydrogen atom written in momentum space is reduced by certain substitutions for the independent and dependent variables to an integral equation for four-dimensional spherical functions having solutions that realize an irreducible representation of the O(4) group. A similar method was used later by Alliluyev, who studied a multidimensional generalization of the Kepler problem (ref. 3), and in recent times independently by Perelomov and Popov (ref. 4), and Kuznetsov (ref. 5) in order to construct the eigenfunctions of the continuous spectrum. However, we believe that this approach cannot be considered satisfactory, since it leads to limiting only one of the possible systems of eigenfunctions of the discrete and continuous spectra.

*Numbers given in margin indicate pagination in original foreign text.

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To appear in the journal Nuclear Physics (Yadernaya Fizika).

The purpose of this article is as follows:

1. To construct a differential equation for the n-dimensional Coulomb problem in momentum space. To show the equivalence of this equation to the wave equation in an n-dimensional Riemannian space of constant curvature $K>0$ (the discrete spectrum) or $K<0$ (the continuous spectrum). To find the generators of the corresponding group of the equation (for example, the motion groups of three-dimensional space of constant positive curvature for the discrete spectrum of the hydrogen atom, and the homogeneous Lorentz group in the case of the continuous spectrum) by solving Killing's equations.

2. To study possible sets of quantum numbers in the three-dimensional case. An article by Smorodinskiy and the author (ref. 6) contains constructions of all possible sets of permutation operators in three-dimensional spaces of constant curvature $K \gtrless 0$. It will be shown that in the case of the hydrogen atom, /4 these operations are expressions which are quadratic in infinitesimal group operators, i.e. components of the Runge-Lenz vector and the angular momentum.

3. To construct normalized systems of eigenfunctions of the continuous and discrete spectra in a number of coordinate systems.

The present article is an attempt to achieve a more general approach to the problem and naturally contains some known results. In this case, it is necessary to note that the symmetry group under consideration, for example O(4) for the hydrogen atom, is not a unique group whose generators correspond to the integrals of motion of the Kepler problem. It turns out that the SU(3) group can also be regarded, in a certain sense, as a symmetry group of the hydrogen atom. In this case, the generators of the SU(3) group are connected with the generators of the O(4) group since they are constructed from components of the Runge-Lenz vector and the angular momentum.[1]

2. The Multidimensional Coulomb Problem in Momentum Space

Let us consider the multidimensional Coulomb problem in momentum space. The wave function $\Psi(p)$ in the p-representation is defined as the Fourier transform of the wave function $\Psi(x)$ in x-representation. Instead of solving the Schrödinger equation in coordinate space with subsequent calculation of the Fourier transform, it is sometimes more convenient to write the Schrödinger equation in the form of an integral equation in momentum space (refs. 2-5). Another possible method is the use of momentum representation of quantum-mechanical operators in which x^i is replaced by $i\partial/\partial p^i$ and $\partial/\partial x^i$ by ip^i. In this case, the Schrödinger equation will have the form of a differential equation in momentum space.

[1] A communication concerning the SU(3) group as a symmetry group of the Kepler problem was presented by the author at the Sixth All-Union Conference on the Physics of Elementary Particles at Uzhgorod on 19 October 1965. This problem will be discussed in detail elsewhere.

We shall write the Schrödinger equation for the n-dimensional Coulomb problem

$$\tfrac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\psi}{(\partial x^i)^2}+\left(E+\frac{\epsilon}{r}\right)\psi=0\ . \tag{2.1}$$

where $r=\left[\sum_{i=1}^{n}(x^i)^2\right]^{\frac{1}{2}}$; $\epsilon=1$ in a field of attraction and $\epsilon=-1$ in a field of repulsion; the constants m, e, and h are set equal to one, in the following form

$$(p^2+p_0^2)\psi=\frac{2\epsilon}{r}\psi\quad,\quad p_0^2=-2E\ . \tag{2.2}$$

Multiplying both sides of the left equation of (2.2) by r, we obtain

$r(p^2+p_0^2)\psi=2\epsilon\psi$, from which

$$r(p^2+p_0^2)r(p^2+p_0^2)\psi=4\epsilon^2\psi\ . \tag{2.3}$$

It is obvious that in the case of scattering (E>0), the eigenfunctions in a field of attraction (solutions of equation (2.2) when $\epsilon=1$) and in a field of repulsion ($\epsilon=-1$) are solutions of the last equation. Making use of the permutation relationships

$$[r,p^2]=2i(\vec{p}\vec{r})\frac{1}{r}-(n-1)\frac{1}{r}\ ;\ [r^2,p^2]=4i(\vec{p}\vec{r})-2n;\ [(\vec{p}\vec{r}),p^2]=2ip^2\ , \tag{2.4}$$

implied by the permutation relationships between the components of an n-dimensional momentum and the coordinates $[p^k,x^l]=i\delta^{kl}(k,l=1,\ldots,n)$, we in turn transform equation (2.3)

$$[(p^2+p_0^2)r^2(p^2+p_0^2)+2i(\vec{p}\vec{r})(p^2+p_0^2)-(n-1)(p^2+p_0^2)]\psi=4\psi\ , \tag{2.5}$$

$$[(p^2+p_0^2)^2r^2+6i(p^2+p_0^2)(\vec{p}\vec{r})-(3n-1)(p^2+p_0^2)-4p^2]\psi=4\psi\ . \tag{2.6}$$

In the p-representation the last equation will be:

$$(p^2+p_0^2)^2\sum_{i=1}^{n}\frac{\partial^2\Psi}{(\partial p^i)^2}+6(p^2+p_0^2)\sum_{i=1}^{n}p^i\frac{\partial\Psi}{\partial p^i}+[(3n-1)(p^2+p_0^2)+4p^2+4]\Psi=0. \tag{2.7}$$

In the next section we shall show the equivalence of equation (2.7) to the wave equation in n-dimensional space of constant curvature.

3. Symmetry of the Equation in Momentum Space

We shall write equation (2.7) with the following notation:

$$F[\Psi] \equiv a^{ij}\Psi_{ij} + b^{i}\Psi_{i} + c\Psi = 0 \quad , \tag{3.1}$$

where $\Psi_{ij} = \partial^2\Psi/\partial p^i \partial p^j$, $\Psi_i = \partial\Psi/\partial p^i$,

and the coefficients $a^{ij} = (p^2 + p_0^2)^2 \delta^{ij}$, $b^i = 6(p^2 + p_0^2)p$, $c = (3n-1)(p^2 + p_0^2) + 4p^2 + 4$ (3.2)

depend on p. We shall consider equation (3.1) in terms of the Riemannian space V_n associated with this equation, whose basic tensor a_{ij} is defined /6

by the equations $a_{jk}a^{ij} = \delta^i_k$; i,k=1,...,n (ref. 7). It is obvious that the metric of the space associated with equation (2.7) is

$$ds^2 = a_{ij}\,dp^i\,dp^j = (p^2 + p_0^2)^{-2} \sum_{i=1}^{n} (dp^i)^2 . \tag{3.3}$$

The symmetry of the Schrödinger equation in the n-dimensional Riemannian space V_n' was investigated in a paper by Smorodinskiy and the author (ref. 1), and it was shown that the group of this equation is the motion group of space V_n' conformal to V_n. In this case, the equation can be reduced to the invariant form of the Schrödinger equation in the V_n' space with a potential proportional to the scalar curvature R of this space. The application of the approach employed in reference 1 is not restricted to the Schrödinger equation and leads to an entirely analogous result for equation (2.7). In this section we shall show the equivalence of equation (2.7) to the Schrödinger equation with the constant potential R=const., i.e. to the wave equation in a space of constant curvature. In the next section we shall construct the Lie algebra of the group permitted by equation (2.7) by solving Killing's equations. In accordance with reference 1, we calculate the quantities

$$a^i = b^i + a^{kl}\Gamma^i_{kl} \qquad (i = 1,\ldots,n) , \tag{3.5}$$

$$H = -2c + a^i_{,i} + \tfrac{1}{4}a^i a_i + \frac{n-2}{2(n-1)}R \quad , \tag{3.4}$$

where R is the scalar curvature of the space with the metric (3.3) associated with equation (2.7), Γ^i_{kl} are Christoffel symbols of the second kind, and the comma denotes the covariant derivatives with respect to the basic tensor a_{ij}. It is convenient to calculate a'^i and H' for the equation

$$F'[\Psi] \equiv e^{-\theta}F[\Psi] \equiv (p^2 + p_0^2)^{-2}F[\Psi] . \tag{3.6}$$

obviously possessing a Euclidean associated space, then proceed to a^i and H, employing the following formulas (refs. 1 and 7):

$$a^i = (a'^i + \frac{n-2}{2} a'^{ij}\theta_j)e^{\theta}, \quad (i=1,\ldots,n); \; H=(p^2+p_0^2)^2 H', \tag{3.7}$$

where $\theta_j=\partial\theta/\partial p^i$. For equation (3.6), formulas (3.4) and (3.5) yield ($a_i^j=a'^{ij}$).

$$a'^{ij} = b'^{ij} = \frac{6p^i}{p^2+p_0^2}, \quad H' = -\frac{2c}{(p^2+p_0^2)^2} + \frac{36p^2}{(p^2+p_0^2)^3} + \frac{6n(p^2+p_0^2)-12p^2}{(p^2+p_0^2)^3}.$$

Making use of (3.7) and (3.2), we obtain /7

$$a^i = \frac{n+1}{2}\frac{\partial}{\partial p^i}(p^2+p_0^2)^2, \tag{3.8}$$

$$H = 2p_0^2 - 8 = \text{const}. \tag{3.9}$$

We shall show that with the aid of the transformation

$$\bar{F}[\Psi] \equiv e^{-\nu} F[e^{\nu}\Psi] \tag{3.10}$$

we can obtain the equation $\bar{F}[\Psi]=0$ (equivalent to the equation $F[\Psi]=0$ in which all $\bar{a}^{ij}=0$. It is not difficult to see that equations (3.1) and (3.10) possess one associated space. Therefore $\bar{a}^i = a^i + 2a^{ij}\nu_j = a^i + 2(p^2+p_0^2)^2\partial\nu/\partial p^i$. Making use of (3.8), we write the system of equations $\bar{a}^i=0$, i=1, ...,n in the form

$$\frac{n+1}{2}\frac{\partial}{\partial p^i}(p^2+p_0^2)^2 + 2(p^2+p_0^2)^2\frac{\partial\nu}{\partial p^i} = 0,$$

from which

$$\frac{\partial\nu}{\partial p^i} = \frac{\partial}{\partial p^i}\ln(p^2+p_0^2)^{-\frac{n+1}{2}}$$

Thus, if we carry out transformation (3.10) with the function

$$e^{-\nu} = (p^2+p_0^2)^{\frac{n+1}{2}}, \tag{3.11}$$

for equation $\bar{F}[\Psi]=0$ the quantities $\bar{a}^{ij}=0$. We note that the quantity H does not vary with transformation (3.10) (ref. 7), i.e. $\bar{H}$ for equation (3.10) is equal to H of (3.9).

We now note the following fact. Condition $\bar{a}^{ij}=0$ and the definition (3.4) of these quantities give $\bar{b}^{ij}=-a^{kl}\Gamma_{kl}^i$ for the coefficients of equation (3.10) (we recall that equations (3.1) and (3.10) possess one associated space), and we can write equation (3.10) in the form:

$$\bar{F}[\Psi] \equiv a^{ij}\Psi_{ij} - a^{kl}\Gamma_{kl}^i\Psi_i + \bar{c}\Psi \equiv \Delta_2\Psi + \bar{c}\Psi = 0. \tag{3.12}$$

where Δ_2 is the second Beltrami differential parameter or Laplacian operator defined on the metric (3.3)

$$\Delta_2 \Psi \equiv a^{ij} (\partial^2 \Psi / \partial p^i \partial p^j - \Gamma_{ij}^k \partial \Psi / \partial p^k) . \tag{3.13}$$

The Riemannian space actually associated with the equation is introduced so /8 that the coefficients of the second derivatives will coincide with the corresponding coefficients of the Laplacian operator (3.13) defined on this space. With the help of transformation (3.10) and $e^{-\nu}$ of (3.11), it is possible to reduce equation (3.1) to the form (3.12), that is, to represent the terms containing derivatives to the form of the Laplacian operator on the metric (3.3).

According to (ref. 1), the coefficient $\bar{c}$ of equation (3.12) is equal to

$$\bar{c} = -\frac{H}{2} + \frac{n-2}{4(n-1)} R = -p_0^2 + 4 + \frac{n-2}{4(n-1)} R . \tag{3.14}$$

where the value of H from (3.9) is substituted. On the other hand, $\bar{c}$ can be calculated in formula $\bar{c} = c + b^i \nu_i + a^{ij} \nu_{ij} + a^{ij} \nu_i \nu_j$ (ref. 7), where $\nu_i = \partial \nu / \partial p^i$, $\nu_{ij} = \partial^2 \nu / \partial p^i \partial p^j$. Substituting ν from (3.11) here, we obtain

$$\bar{c} = -(n-1)^2 p_0^2 + 4 . \tag{3.15}$$

A comparison of (3.14) and (3.15) yields the following expression for R:

$$R = -4n(n-1) p_0^2 = \text{const} . \tag{3.16}$$

Of course, this result could have been obtained directly by calculating the scalar curvature R of the space with metric (3.3). As is known from Riemannian geometry (ref. 8), the scalar curvature R is equal to R=n(1-n)K for a space of constant curvature K. From this,

$$K = 4 p_0^2 . \tag{3.17}$$

Thus, equation (2.7) for the multidimensional Coulomb problem is equivalent to the equation for the eigenvalues of the Laplacian operator

$$\bar{F}[\Psi] \equiv \Delta_2 \Psi - [(n-1)^2 p_0^2 - 4] \Psi = 0 \tag{3.18}$$

in a space of constant curvature K>0 for the discrete spectrum ($p_0^2>0$) and K<0 for the continuous spectrum $p_0^2<0$ (scattering).

Henceforth, when constructing systems of the eigenfunctions of equation (3.18), it will be more convenient to deal with the equation for the eigenvalues of the Laplacian operator in the space of the same constant curvature $K=\pm 1$. In order to obtain such an equation, we divide (3.18) through by $|4p_0^2|$. Then, for the discrete spectrum ($p_0^2>0$)

$$(4p_0^2)^{-1} \bar{F}[\Psi] \equiv \Delta_2 \Psi - [(\tfrac{n-1}{2})^2 - \frac{1}{p_0^2}] \Psi = 0 , \tag{3.19}$$

where Δ_2 is the Laplacian operator on the metric (compare (3.3)) /9

$$ds^2 = \frac{4p_0^2\sum_{q=1}^{n}(dp^i)^2}{(p^2+p_0^2)^2} = \frac{\sum_{i=1}^{n}(dP^i)^2}{(1+\frac{1}{4}P^2)^2} . \tag{3.20}$$

Here $P^i=2p^i/P_0$, $p_0=\sqrt{-2E}$. For the continuous spectrum ($p_0^2<0$)

$$(-4p_0^2)^{-1}\bar{F}[\Psi] \equiv \Delta_2\Psi-[\frac{1}{p_0^2}-(\frac{n-1}{2})^2]\Psi=0 , \tag{3.21}$$

where Δ_2 is the Laplacian operator on the metric

$$ds^2 = \frac{-4p_0^2\sum_{i=1}^{n}(dp^i)^2}{(p^2+p_0^2)^2} = \frac{\sum_{i=1}^{n}(dP^i)^2}{(1-\frac{1}{4}P^2)^2} . \tag{3.22}$$

Here $P^i=2p^i/|p_0|=2p^i/\sqrt{2E}$, $E>0$. Expressions (3.20) and (3.22) are the known Riemannian form of the linear element for spaces of constant curvature K=1 and K=-1, respectively (ref. 8).

As is known (see ref. 9), the eigenvalues of the Laplacian operator on an n-dimensional sphere (i.e., an n-dimensional space of constant positive curvature) are equal to -N(N+n-1), where N is a non-negative whole number. Making use of (3.19), we obtain the well known result (ref. 3):

$$E_N = -\tfrac{1}{2}(N+\tfrac{n-1}{2})^{-2} . \tag{3.23}$$

The eigenvalues of the Laplacian operator in a space of constant negative curvature (the continuous spectrum) are equal to N(N+n-1), where N is any complex number (see ref. 9). In order for N(N+n-1) to be real, it is necessary for either N to be real or for N=-(n-1)/2+ik, where k is real. From (3.21) we obtain $E_N=\frac{1}{2}(N+\frac{n-1}{2})^{-2}$. It is obvious that the case of real N does not apply to the continuous spectrum ($E>0$). When N=-(n-1)/2+ik, we have:

$$E_k = \frac{1}{2k^2} . \tag{3.24}$$

When n=3, these eigenvalues correspond to unitary representations of a Lorentz group. In this case, real N corresponds to an additional series of representations, and N=1+ik to representations of the basic series (ref. 10). The latter are also realized by the eigenfunctions of the continuous spectrum corresponding to values of the principal quantum number (3.24). The following part of the article is devoted to finding the possible sets of other quantum numbers such that the group properties of equations (2.7) and (3.18) (groups permitting equivalent equations are similar) are important to the construction of these numbers.

4. Killing's Equations and the Lie Algebra

The symmetry group of an equation of the form (3.1) is understood to be the set of those transformations of independent and dependent variables with a closed operator algebra

$$X = \xi^i \partial/\partial p^i + \sigma\Psi\partial/\partial\Psi \,. \tag{4.1}$$

with respect to which the following equation is invariant

$$XF[\Psi] = \lambda F[\Psi] \tag{4.2}$$

The operators X are extended to second-order derivatives with respect to $\Psi;\xi^i;\sigma$ and the factor λ depend on p (ref. 7). These functions are found from a system of determining equations which are reduced to Killing's equations when $a^i \equiv 0$:

$$\xi_{i,j} + \xi_{j,i} = 0 \qquad (i,j = 1,\ldots,n) \tag{4.3}$$

which are the determining equations of the motion group of some Riemannian space (ref. 1). In the case of equation (3.18), this space coincides with the associated space since for it H=const. according to (3.9). Thus, the symmetry group of equation (3.18) and equation (2.7) equivalent to it is the motion group of a space of constant curvature K>0 for the discrete spectrum E<0 and K<0 for the continuous spectrum E>0 (refer to (3.20 and (3.22)). As shown in reference 1, on the condition that $a^i=0$, the operators (4.1) are of the form $\bar{X}=\xi^i\partial/\partial p^i$ and transform the solutions of equation (3.18) to the solutions, that is, to the manifold $\bar{F}=0$

$$[\bar{X},\bar{F}] = 0 \,. \tag{4.4}$$

In this case it is obvious that if Ψ is a solution of equation (2.7),

$$\bar{\Psi} = e^{-\nu}\,\Psi = (p^2 + p_0^2)^{\frac{n+1}{2}}\Psi \tag{4.5}$$

is a solution of equation (3.10).

Let us go on to solving Killing's equations (4.3) which determine the equation of the motion group of the space of constant curvature $K=4p_0^2$ (3.17) with the metric (3.3). We can verify with a direct check that the general solution of equations (4.3) contains $\frac{n(n+1)}{2}$ parameters and has the form:

$$\xi^i = \frac{a_i}{2|p_0|}[2(p^i)^2 - p^2 + p_i^2] + \sum_{s=1}^{n}{}' \frac{a_s}{|p_0|} p^s p^i + \sum_{s=1}^{n}{}' \beta_{is} p^s \,, \quad (i=1,\ldots,n) \,. \tag{4.6}$$

where the prime with the summation sign indicates that it is necessary to omit the term for s=1. The constants α_i are arbitrary and the constants $\beta_{is}=-\beta_{si}$. We shall write the operators corresponding to the solution (4.6). /11

The constants β_{is} yield the ordinary angular momentum operators:

$$\bar{X}_{ij} = p^i \partial/\partial p^j - p^j \partial/\partial p^i \quad (i,j = 1,\ldots,n). \tag{4.7}$$

and the constants α_i lead to the operators $\bar{X}_{0i}=-\bar{X}_{i0}$ (i=1, ..., n).

$$\bar{X}_{0i} = \frac{1}{2|p_0|}\left[2(p^i)^2 - p^2 + p_0^2\right]\frac{\partial}{\partial p^i} + \frac{p^i}{|p_0|}\sum_{\alpha=1}^{n} p^\alpha \frac{\partial}{\partial p^\alpha} - \sum_{\alpha=1}^{n}\frac{1}{|p_0|}\left(p^i p^\alpha + \delta_{i\alpha}\frac{p_0^2 - p^2}{2}\right)\frac{\partial}{\partial p^\alpha}.$$

The operators obtained when $p_0^2>0$ ($E<0$ is the discrete spectrum) satisfy the commutative relationships

$$[\bar{X}_{\lambda\mu}, \bar{X}_{\rho\sigma}] = \delta_{\lambda\sigma}\bar{X}_{\mu\rho} + \delta_{\mu\rho}\bar{X}_{\lambda\sigma} - \delta_{\lambda\rho}\bar{X}_{\mu\sigma} - \delta_{\mu\sigma}\bar{X}_{\lambda\rho} \quad (\lambda,\mu,\sigma,\rho = 0,1,\ldots,n). \tag{4.9}$$

which coincide with the permutation relationships of the Lie algebra for the 0(n+1) group. In the case of the continuous spectrum ($p_0^2<0$; $E>0$), the corresponding group is the 0(1,n) group (when n=3, the Lorentz group with one time-like dimension). Making use of the notation $\bar{X}_{ij}=-\epsilon_{ijk}\bar{L}_k$, we have ($p_0^2<0$):

$$\begin{gathered}[\bar{L}_i, \bar{L}_k] = \epsilon_{ikm}\bar{L}_m, \quad [\bar{L}_i, \bar{X}_{0k}] = \epsilon_{ikm}\bar{X}_{0m}, \\ [\bar{X}_{0i}, \bar{X}_{0k}] = -\epsilon_{ikm}\bar{L}_m.\end{gathered} \tag{4.10}$$

The operators $\bar{X}$ form the basis of the Lie algebra for the group of equation (3.10) $\bar{F}[\Psi]=0$. From these operators it is not difficult to construct the basis X of the Lie algebra of the equivalent equation (3.1) $F[\Psi]=0$. According to (4.4), the operators $\bar{X}$ of (4.7) and (4.8) are interchangeable with the "Hamiltonian" $\bar{F}$ on the manifold of solutions $\bar{\Psi}$ of equation (3.10), that is, $[\bar{X},\bar{F}]\bar{\Psi}=0$. We shall prove that the operators

$$X = e^{\nu}\bar{X}e^{-\nu} = (p^2 + p_0^2)^{-\frac{n+1}{2}}\bar{X}(p^2 + p_0^2)^{\frac{n+1}{2}} \tag{4.11}$$

are interchangeable with the operator $F=e^{\nu}\bar{F}e^{-\nu}$ (see (3.10)) on the manifold Ψ /12 of solutions of equation (3.1) $F[\Psi]=0$. Indeed

$$[X, F]\Psi = [e^{\nu}\bar{X}e^{-\nu}, e^{\nu}\bar{F}e^{-\nu}]\Psi = e^{\nu}[\bar{X},\bar{F}]e^{-\nu}\Psi = e^{\nu}[\bar{X},\bar{F}]\bar{\Psi} = 0,$$

for according to (4.5) if Ψ is a solution of equation (3.1), $e^{-\nu}\Psi$ is a solution of (3.10), i.e., $e^{-\nu}\Psi=\bar{\Psi}$. Applying (4.11) to the operators $\bar{X}$ (4.7) and (4.8), we obtain the basis X

$$X_{ij} = \bar{X}_{ij} = p^i \partial/\partial p^j - p^j \partial/\partial p^i = iL_{ij} \quad (i,j = 1,\ldots,n), \tag{4.12}$$

$$X_{0i} = (p^2 + p_0^2)^{-\frac{n+1}{2}}\bar{X}_{0i}(p^2 + p_0^2)^{\frac{n+1}{2}} = \frac{1}{|p_0|}\left(p^i\sum_{\alpha=1}^{n} p^\alpha\frac{\partial}{\partial p^\alpha} + \frac{p_0^2 - p^2}{2}\frac{\partial}{\partial p^i} + \frac{n+1}{2}p^i\right) = -\frac{iA_i}{|p_0|} \tag{4.13}$$

where A is the multidimensional generalization of the Runge-Lenz vector

$$A_i = \tfrac{1}{2}\sum_{\alpha=1}^{n}(L_{\alpha i}p^{\alpha} + p^{\alpha}L_{\alpha i}) + \frac{\epsilon x^i}{r} = L_{\alpha i}p^{\alpha} - \frac{i(n-1)}{2}p^i + \frac{\epsilon x^i}{r} =$$

$$= p^i(\vec{p}\vec{r}) - p^2x^i + i\frac{n-1}{2}p^i + \frac{\epsilon x^i}{r} = p^i(\vec{p}\vec{r}) - \frac{p^2x^i}{2} + i\frac{n+1}{2}p^i - x^iH\ ; \qquad (4.14)$$

$$H = \frac{p^2}{2} - \frac{\epsilon}{r} \qquad (i = 1, \ldots, n).$$

We recall that $\epsilon=1$ for the bound states; for the continuous spectrum, $\epsilon=1$ corresponds to scattering in a field of attraction, and $\epsilon=-1$ corresponds to scattering in a field of repulsion. The operators (4.12) and (4.13) form the basis of the Lie algebra for the symmetry group of equation (3.1). Making use of (4.11), we obtain from (4.9) the commutative relationships in terms of the angular momentum operators $L_{ij}=-iX_{ij}$ and the quantities $A'_i = A_i/\sqrt{2|H|} = iX_{0i}$ $(i=1,\ldots,n)$. In the case of the discrete spectrum, the corresponding expressions for the operators $X_{\rho\sigma}(\rho,\sigma=0,1,\ldots,n)$ coincide with the relationships (4.9) whose right sides were multiplied by -i and were obtained by Bargmann (ref. 11) for n=3 (the hydrogen atom). Relationships analogous to (4.9) were constructed recently by D'yerdi and Revan (ref. 12). In the case of the continuous spectrum, the commutative relationships of the components of the vector A'_i and of the angular momentum L_{ij} also coincide with (4.10), whose right side is multiplied by i. After deriving formulas (4.11)-(4.13) which establish the connection between the bases X and $\overline{X}$ of the Lie algebra for the symmetry group admitted by equations (3.1) and (3.10), we can go on directly to calculating the quantum numbers of the problem.

5. Possible Sets of Quantum Numbers

In this section we shall consider sets of operators which are diagonal on systems of eigenfunctions corresponding to separation of the variables of equation (3.10) or (3.18) in different coordinate systems. It was shown in Section 3 that equation (3.10) is the equation of eigenvalues of the Laplacian operator in a space of constant curvature (3.18). Possible sets of operators are constructed in reference 6 which are commutative with the Laplacian and with each other in all coordinate systems in which the variables of the equation $\Delta_2\Psi+\lambda\Psi=0$ are separated in three-dimensional spaces of constant curvature $K\gtrless0$ (the constants of separation are the eigenvalues of these operators). These operators are transformed to operators corresponding to equation (3.1) with the aid of relationships (4.11)-(4.13).

It is known that the motion group of space V_n of constant curvature $K>0$ can be regarded as rotation about the origin of a Cartesian coordinate system in a convolute flat space S_{n+1} (ref. 8). The Cartesian coordinates of points on the hypersurface V_n in an (n+1)-dimensional convolute space S_{n+1} are the

Weierstrass coordinates $ds^2 = \frac{\sum_{i=1}^{n}(dp^i)^2}{(p^2+p_0^2)^2}$ for a space with the metric $\xi_i = f_i(p^1,\ldots,p^n)$, $\frac{1}{\sqrt{K}}\xi_0 = f_0(p^1,\ldots,p^n)$ and can be determined with the aid of the following equations (ref. 8):

$$\sum_{i=1}^{n}(d\xi_i)^2 + \frac{1}{K}d\xi_0^2 = \sum_{i=1}^{n}(dp^i)^2/(p^2+p_0^2)^2 \; ; \xi^2 + \frac{1}{K}\xi_0^2 = \frac{1}{K} \, .$$

$$\sum_{i=0}^{n}\frac{\partial f_i}{\partial p^j}\frac{\partial f_i}{\partial p^k} = 0 \quad (j \neq k) \; ; \; \sum_{i=0}^{n}\left(\frac{\partial f_i}{\partial p^j}\right)^2 = (p^2+p_0^2)^{-2} \quad (j=1,\ldots,n) \, . \tag{5.1}$$

We shall express the operators $\overline{X}$ (4.7) and (4.8) by means of infinitesimal angular momentum operators in a convolute flat space. Let us consider the cases of discrete and continuous spectra separately in the three-dimensional case.

The Discrete Spectrum

The Weierstrass coordinates in a space of constant curvature K=1 with metric (3.20) are

$$\xi^2 + \xi_0^2 = 1 \, , \quad \xi_i = P^i/(1+\tfrac{1}{4}P^2) \, , \quad \xi_0 = (1-\tfrac{1}{4}P^2)/(1+\tfrac{1}{4}P^2) \quad (i=1,\ldots,n).$$

It is not difficult to verify that these quantities satisfy conditions (5.1). Since P^i in (3.20) is $P^i = 2p^i/p_0$,

$$\xi_i = 2p_0 p^i/(p^2+p_0^2) \, , \quad \xi_0 = (p_0^2-p^2)/(p_0^2+p^2) \quad (i=1,\ldots,n) \, . \tag{5.2}$$

and naturally coincide with the coordinates introduced by Fok (ref. 2). /14
The connection between the operators $\overline{X}$ of (4.7) and (4.8) and the operators of the four-dimensional moment (generators of rotation groups) in a convolute Euclidean space $M_{\rho\sigma} = \xi_\rho \partial/\partial\xi_\sigma - \xi_\sigma \partial/\partial\xi_\rho$ is known (refer, for example, to ref. 11):

$$\overline{X}_{\rho\sigma} = M_{\rho\sigma} \quad (\sigma,\rho = 0,1,\ldots,n) \, . \tag{5.3}$$

In three-dimensional Riemannian space of constant curvature K>0, there exist six coordinate systems in which the variables of Laplace's equation are separated in accordance with six possible sets of quantum numbers (ref. 6). We shall consider two coordinate systems for which the corresponding systems of eigenfunctions realizing representations of class 1 of the O(4) group will be given in the next section.

(a) The cylindrical coordinate system is given by the following formulas:

$$\begin{aligned} \xi_0 &= \cos\theta\cos\phi_1 \, , \quad \xi_2 = \sin\theta\cos\phi_2 \, , \\ \xi_1 &= \cos\theta\sin\phi_1 \, , \quad \xi_3 = \sin\theta\sin\phi_2 \, . \end{aligned} \tag{5.4}$$

We shall express the operators $\overline{X}_2 = \partial^2/\partial\phi_1^2$ and $\overline{X}_3 = \partial^2/\partial\phi_2^2$ which are diagonal in

this representation by the coordinates ξ. Omitting elementary transformation, we obtain

$$\bar{X}_2 = (\xi_0\partial/\partial\xi_1 - \xi_1\partial/\partial\xi_0)^2 = M_{01}^2 ; \quad \bar{X}_3 = (\xi_2\partial/\partial\xi_3 - \xi_3\partial/\partial\xi_2)^2 = M_{23}^2 \tag{5.5}$$

According to formulas (5.3), (4.11)-(4.13), these operators, which are diagonal on solutions of Laplace's equation, correspond to the following operators, which are diagonal on a system of functions that are solutions of the initial equation (3.1):

$$\begin{aligned} X_2 &= (p^2 + p_0^2)^{-2}\bar{X}_2(p_{01}^2 + p_0^2)^2 = X_{01} = -A_1^2/p_0^2 , \\ X_3 &= (p^2 + p_0^2)^{-2}\bar{X}_3(p^2 + p_0^2)^2 = X_{23}^2 = -L_{23}^2 . \end{aligned} \tag{5.6}$$

(b) The spherical system is given by the formulas

$$\begin{aligned} \xi_0 &= \cos\alpha , \quad \xi_1 = \sin\alpha\sin\theta\sin\phi , \\ \xi_2 &= \sin\alpha\cos\theta , \quad \xi_3 = \sin\alpha\sin\theta\cos\phi . \end{aligned} \tag{5.7}$$

We obtain the following diagonal operators in a like manner:

$$X_2 = -L^2 , \quad X_3 = -L_{12}^2 . \tag{5.8}$$

For brevity we shall restrict ourselves to these two coordinate systems for which normalized systems of eigenfunctions will be given in the next section. We note that as in all the other systems, the diagonal operators are also expressions which are quadratic in the infinitesimal operators of the group.

The Continuous Spectrum

The Weierstrass coordinates in a space of constant curvature K=-1 with metric (3.22) are

$$\xi_i = P^i/(1-\tfrac{1}{4}P^2), \quad \xi_0 = (1+\tfrac{1}{4}P^2)/(1-\tfrac{1}{4}P^2) \quad (i=1,\ldots,n) , \tag{5.9}$$

or, bearing in mind that (3.22) $P^i = 2p^i/|p_0|$, and writing $p_0^2 = 2E$ henceforth in formulas for the continuous spectrum, we obtain

$$\xi_i = 2p_0p^i/(p_0^2 - p^2) , \quad \xi_0 = (p_0^2 + p^2)/(p_0^2 - p^2) , \quad p_0 = \sqrt{2E} . \tag{5.10}$$

We shall again represent the operators $\bar{X}$ from (4.7) and (4.8) in the form of generators of the rotation group for the convolute pseudo-Euclidean ξ-space:

$$M_{k\ell} = \xi_k\partial/\partial\xi_\ell - \xi_\ell\partial/\partial\xi_k , \quad M_{0k} = \xi_0\partial/\partial\xi_k + \xi_k\partial/\partial\xi_0 \quad (k,\ell = 1,\ldots,n)$$

(when n=3, the generators of a Lorentz group with one time-like dimension). It

is not difficult to verify that

$$\bar{X}_{k\ell} = M_{k\ell} \quad , \quad \bar{X}_{0k} = -M_{0k} \quad (k, \ell = 1, \ldots n) \, . \tag{5.11}$$

Further calculations are carried out as in the case of the discrete spectrum (complete sets of commutative operators in Lobachevskiy space have been constructed in reference 6). Let us consider the following coordinate systems:

$$\begin{aligned} \xi_0 &= \mathrm{ch}\, a \, \mathrm{ch}\, b \quad , \qquad \xi_2 = \mathrm{sh}\, b \sin\phi \quad , \\ \xi_1 &= \mathrm{sh}\, a \, \mathrm{ch}\, b \quad , \qquad \xi_3 = \mathrm{sh}\, b \cos\phi \end{aligned} \tag{5.12}$$

characterized by the following operators:

$$X_2 = -A_1^2/2H \quad , \quad X_3 = -L_{23}^2 \quad , \tag{5.13}$$

(b) The spherical system S:

$$\begin{aligned} \xi_0 &= \mathrm{ch}\, a \quad , \qquad \xi_2 = \mathrm{sh}\, a \sin\theta \cos\phi \quad , \\ \xi_3 &= \mathrm{sh}\, a \cos\theta \quad , \qquad \xi_1 = \mathrm{sh}\, a \sin\theta \sin\phi \quad , \end{aligned} \tag{5.14}$$

is characterized by the operators

$$X_2 = L^2 \quad , \quad X_3 = L_{12}^2 \, . \tag{5.15}$$

(c) The hyperbolic system H:

$$\begin{aligned} \xi_1 &= \mathrm{sh}\, a \quad , \qquad \xi_2 = \mathrm{ch}\, a \, \mathrm{sh}\, b \sin\phi \quad , \\ \xi_0 &= \mathrm{ch}\, a \, \mathrm{ch}\, b \quad , \qquad \xi_3 = \mathrm{ch}\, a \, \mathrm{sh}\, b \cos\phi \quad . \end{aligned} \tag{5.16}$$

The following operators are diagonal on the eigenfunctions corresponding to separation of variables in a hyperbolic coordinate system:

$$X_2 = (A_2^2 + A_3^2)/2H - L_{23}^2 \quad , \quad X_3 = L_{23}^2 \, . \tag{5.17}$$

(d) The orispherical (sic) system O:

$$\begin{aligned} \xi_0 &= \tfrac{1}{2}[e^{-a} + (r^2+1)e^{a}] \quad , \quad \xi_3 = \tfrac{1}{2}[e^{-a} + (r^2-1)e^{a}] \, , \\ \xi_2 &= re^{a}\cos\phi \quad , \qquad \xi_1 = re^{a}\sin\phi \quad . \end{aligned} \tag{5.18}$$

In this system the following operators are diagonal:

$$X_2 = \left(\frac{A_1}{\sqrt{2H}} - L_{13}\right)^2 + \left(\frac{A_2}{\sqrt{2H}} - L_{23}\right)^2 \quad , \quad X_3 = L_{12}^2 \quad . \tag{5.19}$$

We note that the coordinate systems we have considered, whose coordinate surfaces are transitivity surfaces, can be classified in terms of invariants of

subgroups of the O(4) group, or the O(1,3) group. Together with the second-order Casimir operator, the invariants of the subgroups for complete sets of commutative observables, and their common eigenfunctions form the basis of the representation. This classification, cited in an article by Winternitz and Frisch (ref. 13) for the Lorentz group also remains valid for the basis of the operators L_{ik} and $A_i'=A/\sqrt{2|H|}$. Thus the operators (5.13), (5.15), etc. are invariants of different subgroups of the symmetry group of the problem. After finding the explicit form of the diagonal operators in different coordinate systems, we proceed to constructing normalized eigenfunctions.

6. Eigenfunctions of the Discrete Spectrum

Before we write the normalized eigenfunctions, we shall show that the eigenvalues of the operators X_2 and X_3 coincide with the eigenvalues of the operators $\bar{X}_2$ and $\bar{X}_3$ of (5.5) which are separation constants of the variables in equation (3.18) (ref. 8). Let $X\Psi=\lambda\Psi$. According to (4.8), $\bar{X}e^{-\nu}\Psi=\lambda e^{-\nu}\Psi$. We multiply this expression on the left by e^{ν}. Then $e^{\nu}\bar{X}e^{-\nu}\Psi=X\Psi=\lambda\Psi$ by definition of the operators X (4.11) and since λ is a c number. Thus, the eigenvalues of the operators X and $\bar{X}$ which act on the respective systems Ψ and $\bar{\Psi}$ of eigenfunctions coincide.

The graphical method for constructing normalized eigenfunctions $\bar{\Psi}$ which are solutions of Laplace's equation on a four-dimensional sphere in the coordinate systems (5.4) and (5.7) was proposed by Vilenkin, Kuznetsov, and Smorodinskiy (ref. 14). Making use of the algorithm of reference 14, we obtain the orthonormalized eigenfunctions in system (5.7).

$$\bar{\Psi}_{N\ell m}(\alpha,\theta,\phi)=[N_n^{a,a}]^{-1/2}(1-\cos^2\alpha)^{\frac{\ell}{2}}\varphi_n^{a,a}(\cos\alpha)Y_{\ell m}(\theta,\phi), \tag{6.1}$$

where $\varphi_n^{a,a}(\cos\alpha)$ are Gegenbauer polynomials, $Y_{\ell m}(\theta,\phi)$ are spherical functions, $n=N-\ell$, $a=\ell+\frac{1}{2}$, and $n=0, 1, 2, \ldots$ The quantities $-e(\ell+1)$ and $-m^2$ which are constants of separation (ref. 14) are eigenvalues of the operators X (5.8) acting on solutions Ψ of equation (3.1)

$$\Psi=(p^2+p_0^2)^{-2}\,\bar{\Psi}_{N\ell m}(\alpha,\theta,\phi). \tag{6.2}$$

N is the principal quantum number (3.23), and the norm $N_n^{a,a}$ is equal to

$$N_n^{a,a}=\frac{2^{2a+1}\Gamma^2(n+a+1)}{(2n+2a+1)\Gamma(n+1)\Gamma(n+2a+1)}. \tag{6.3}$$

Formulas (6.1) and (6.3) correspond to the Fok solution (ref. 2).

For system (5.4), we have

$$\bar{\Psi}_{Nk_1k_2}(\theta,\chi_1,\chi_2)=\tilde{D}^{\ell/2}_{k_1,k_2}(\chi_1,2\theta,\chi_2). \tag{6.4}$$

where $\approx{D}^{l/2}_{k_1,k_2}$ is the normalized general spherical Wigner function (ref. 14)

$\chi_1=\phi_1+\phi_2$, $\chi_2=\phi_1-\phi_2$, $k_1=(m_1-m_2)/2$, $k_2=(m_1+m_2)/2$, $-\frac{l}{2}\le k_1, k_2\le\frac{l}{2}$. The constants of separation $-m^2_1$ and $-m^2_2$ are eigenvalues of the operators (5.6); $m_1,m_2=0,\pm1,\pm2,\ldots$ The transformation to the basis of these operators is analogous to (6.2).

7. The Eigenfunctions of the Continuous Spectrum

In the case of the continuous spectrum $E>0$ (scattering), it is necessary to consider a hyperboloid of two sheets instead of a hypersphere; in this case, the momenta $0<p<\sqrt{2E}=p_0$ correspond to the upper field of the hyperboloid and the momenta $p>\sqrt{2E}$ to the lower field (ref. 2). Transformation from one field of the hyperboloid to the other corresponds to inversion in the convolute pseudo-Euclidean space $\xi\to-\xi$. The value of the function $\bar{\Psi}_b(\xi)(\xi_0>1)$ which is a solution of Laplace's equation with an eigenvalue characterizing real k (3.24) on the upper field of the hyperboloid is connected with the value of the function at the conjugate point $\xi'=-\xi$ on the lower field $\bar{\Psi}_H(\xi')(\xi_0<-1)$ by the following relationships (refs. 4 and 5):

$$\bar{\Psi}_b(\xi)/\bar{\Psi}_H(\xi')=c_1/c_2=-e^{\mp\pi k} \quad . \tag{7.1}$$

where $k=p_0^{-1}$. Normalizing c_1 and c_2 with the condition $|c_1|^2+|c_2|^2=1$, we obtain

$$c_1^{\pm}=(1+e^{\mp 2\pi k})^{-1/2} \quad , \quad c_2^{\pm}=-(1+e^{\pm 2\pi k})^{-1/2}, \tag{7.2}$$

and we write the function defined over the entire interval $0<p<\infty$ in the form

$$\bar{\Psi}^{\pm}(\xi)=\begin{cases} c_1^{\pm}\bar{\Psi}_b(\xi) \quad , & p<p_0 \\ c_2^{\pm}\bar{\Psi}_b(-\xi) \quad , & p>p_0 \end{cases} \quad . \tag{7.3}$$

The ambiguity of the solutions obtained here and defined on the entire hyperboloid, corresponding to two possible values $c^{\pm}_{1,2}$ has a simple physical meaning. It can be seen from the derivation cited in section 2 that the eigenfunctions in both the field of attraction ($\varepsilon=1$) and the field of repulsion ($\varepsilon=-1$) are solutions of the equation under consideration. Then, $\bar{\Psi}^+$ corresponds to the first case and $\bar{\Psi}^-$ to the second case. We note that it is obvious that $\bar{\Psi}_k{}^+=\bar{\Psi}_-{}^-{}_k$. Now, in order to normalize the functions determined on a hyperboloic of two sheets, it is sufficient to normalize on field. In references 4 and 5,

the relationships (7.1) and (7.2) were obtained for the eigenfunctions in a spherical coordinate system (6.14). It is natural that these relationships do not depend on the selection of the system of eigenfunctions.

The eigenfunctions of the Laplacian operator on the hyperboloid in systems C, S, H, and O (5.12) and (5.19) were given in an article by Vilenkin and Smorodinskiy (ref. 15). These functions (not normalized) are of the following form in the S system:

$$\bar{\Psi}_{k\ell m}(a,\theta,\phi) = (\operatorname{sh} a)^{-1/2} \mathcal{P}^{-(\ell+1/2)}_{-1/2+ik}(\operatorname{ch} a) \mathcal{P}^{\ell}_{m}(\cos\theta) e^{im\phi} , \tag{7.4}$$

where the constants of separation $l(l+1)$ and m^2 are eigenvalues of the operators (5.15), $\ell = 0, +1, \ldots$, $m = \ell, \ell-1, \ldots, -\ell$.

In the hyperbolic system, the eigenfunctions are equal to

$$\bar{\Psi}_{kqm}(a,b,\phi) = (\operatorname{ch} a)^{-1} \mathcal{P}^{ik}_{-1/2+iq}(\operatorname{th} a) \mathcal{P}^{im}_{-1/2+iq}(\operatorname{ch} b) e^{im\phi} , \tag{7.5}$$

the constants of separation $q^2+1/2$, where q is real and m^2 are eigenvalues of the operators (5.17), and m is a whole number. In the system (5.18)

$$\bar{\Psi}_{k\kappa m}(a,r,\phi) = e^{-a} K_{ik}(\kappa e^{-a}) J_m(\kappa r) e^{im\phi} , \tag{7.6}$$

J_m are Bessel functions and K_{ik} is the MacDonald function.

In the cylindrical system (5.12), the eigenfunctions

$$\bar{\Psi}_{krm}(a,b,\phi) = e^{i(ra+m\phi)} (\operatorname{sh} b)^m (\operatorname{ch} b)^{-m-1-ik} \times F\left(\frac{m+1+ik+ir}{2}, \frac{m+1+ik-ir}{2}, m+1, \operatorname{th}^2 b\right) , \tag{7.7}$$

$F(\alpha,\beta,\gamma,\delta)$ is the hypergeometric function, and $-r^2$ and $-m^2$ are separation constants which are eigenvalues of the operators (5.13).

We have obtained various systems of eigenfunctions for the continuous spectrum in momentum space. Now, it is required to normalize these functions. Reference 15 gives conversion formulas for expansion in eigenfunctions (7.4)-(7.7). Using the S-system as an example, we shall show how to normalize by making use of the conversion formula. The remaining cases are considered in a like manner and are reduced to integrating the formulas derived in reference 15.

Let there be a function $f(\xi)$ given on the hyperboloid $\xi_0^2-\xi^2=1$. Then, in the unitary case N=-1+ik

$$f(\xi) = (2\pi)^{-3/2} \sum_{\ell,m} (-1)^{\ell} \int_0^{\infty} \frac{\Gamma(ik)}{\Gamma(ik-\ell)} \frac{{}^{*}\ell_m(k)}{\sqrt{\operatorname{sh} a}} \mathcal{P}^{-(\ell+1/2)}_{-1/2+ik}(\operatorname{ch} a) Y_{\ell m}(\theta,\phi) k^2 dk . \tag{7.8}$$

The formulas for inverse transformation are of the form:

$${}^{*}\ell_m(k) = \frac{(-1)^{\ell} (2\pi)^{3/2} \Gamma(-ik)}{\Gamma(-ik-\ell)} \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} f(\xi) \frac{1}{\sqrt{\operatorname{sh} a}} \mathcal{P}^{-(\ell+1/2)}_{-1/2-ik}(\operatorname{ch} a) Y^{*}_{\ell m}(\theta,\phi) \frac{d^3\xi}{\xi_0} . \tag{7.9}$$

where $k = \frac{1}{p_0} = \frac{1}{\sqrt{2E}}$, $l(l+1)$ (ref. 15). It follows from (7.8) and (7.9) that the function

$$\bar{\Psi}_{k\ell m}(a,\theta,\phi) = \frac{|\Gamma(ik)|\, k}{|\Gamma(ik-\ell)|} \mathcal{P}^{-(\ell+1/2)}_{-1/2+ik}(\operatorname{ch} a) \frac{1}{\sqrt{\operatorname{sh} a}} Y_{\ell m}(\theta,\phi) \tag{7.10}$$

satisfies the normalizing condition

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \bar{\Psi}_{k\ell m}(a,\theta,\phi) \bar{\Psi}^*_{k'\ell' m'}(a,\theta,\phi) \frac{d^3\xi}{\xi_0} = \delta(k-k')\delta_{\ell\ell'}\delta_{mm'} \tag{7.11}$$

It then follows from (6.2), (7.2), (7.3), and (7.10) that

$$\Psi^{(\pm)}_{k\ell m}(a,\theta,\phi) = (p^2 - p_0^2)^{-2} \frac{\Gamma(ik)}{\Gamma(ik-\ell)} \left|\frac{2p_0 p}{p_0^2 - p^2}\right|^{-1/2} \mathcal{P}^{-(\ell+1/2)}_{-1/2+ik}\left|\frac{p^2+p_0^2}{p^2-p_0^2}\right| Y_{\ell m}(\theta,\phi) \begin{cases} c_2^{(\pm)}, & p < p_0 \\ c_2^{(\pm)}, & p > p_0 \end{cases} \tag{7.12}$$

where $k = \frac{1}{p_0} = 1\sqrt{2E}$, $l(l+1)$, and m^2 are eigenvalues of the operators (5.15): L_{12}^2 and L^2. The functions (7.5)-(7.7) are normalized in a like manner.

The author thanks Ya. A. Smorodinskiy for his interest in this work and discussing the results that were obtained.

Received 22 March 1966

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Translation for National Aeronautics and Space Administration
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